Exotic group algebras, crossed products, and coactions

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Quantum Groups Seminar Copenhagen/Trondheim,
November 8, 2022

Exotic group algebras

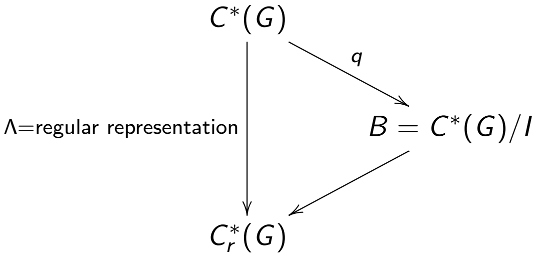
Given a locally compact group G, then the C^*-algebras $C_r^*(G)$ defined by the regular representation of G over $L^2(G)$ and $C^*(G)$ defined by the universal representation of G are of special interest.

In his seminal 1964 paper P. Eymard noted that there are algebras in between. Until recently there have been little interest in studying these algebras, but work of Brown-Guentner caught our interest.

For quantum groups similar questions also was studied by Bedos-Murphy-Tuset and Kyed-Soltan. (And maybe others?)

Exotic group algebras

So we shall look at C^*-algebras B between $C_r^*(G)$ and $C^*(G)$, i.e.



where $I = \ker q$ is an ideal in $C^*(G)$.
We call B *exotic* if

$$\{0\} \subsetneq I \subsetneq \ker \Lambda.$$

We shall also need the the dual spaces, the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$, $B_r(G) = C_r^*(G)^*$ and

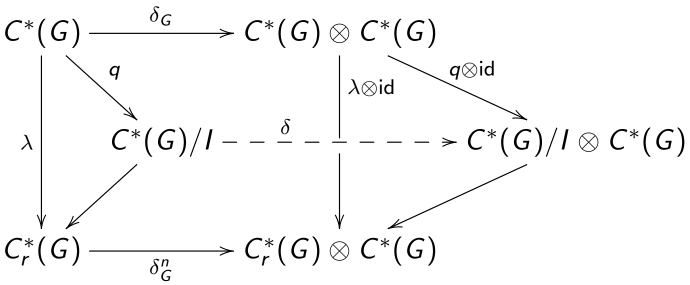
$$E = B^* = I^\perp = \{\varphi \in B(G) \mid \varphi(I) = \{0\}\}.$$

These are G-invariant subspaces of $B(G)$ with

$$B_r(G) \subset E \subset B(G) \subset UCB(G).$$

It seems hopeless to look at all these spaces.

Look at the diagram (for G discrete)



Here δ_G and δ^n_G sends g to $g \otimes g$.
Then $I \subset \ker \lambda$, and:

Theorem

A large quotient $C^*(G)/I$ carries a coaction if and only if the annihilator $E = I^\perp$ in the Fourier-Stieltjes algebra $B(G) = C^*(G)^*$ is a G-invariant ideal.

Large ideals

Thus, large quotients $C^*_E(G) = C^*(G)/{}^\perp E$ of $C^*(G)$ carrying coactions are classified by **large ideals** E of $B(G)$, i.e., G-invariant weak*-closed nonzero ideals (which then automatically contain the reduced Fourier-Stieltjes algebra $B_r(G) = C^*_r(G)^*$).

It appears that there are **lots** of these "exotic ideals":

Definition

Let E_p be the weak*-closure in $B(G)$ of $\text{span}\{P(G) \cap L^p(G)\}$, where $P(G)$ denotes the set of positive type functions on G.

Theorem

E_p is a large ideal in $B(G)$.

Theorem (Godement, Carey)

For $1 \leq p \leq 2$, $E_p = B_r(G)$.

Theorem (Higson, Okayasu, Ozawa, Brown-Guentner)

For $2 \leq p < q < \infty$ *and* $G = \mathbb{F}_2$, $E_p \neq E_q$.

Theorem (Wiersma)

For $2 \leq p < q < \infty$ *and* $G = SL(2, \mathbb{R})$, $E_p \neq E_q$.

Note: For semi-simple Lie groups (square) integrable representations are of special interest.

How do we get large ideals in $B(G)$?

Theorem (Brown-Guentner)

Let D be G-invariant ideal of $CB(G)$.
Then $E = w^*$-closure of $\operatorname{span}\{P(G) \cap D\}$ is a large ideal in $B(G)$.

Question

Do all large ideals in $B(G)$ arize this way?
If so the correspondance will not be 1-1.

Example (Haagerup)

The Rajchman algebra $B_0(G) = B(G) \cap C_0(G)$ is a norm-closed G-invariant ideal in $B(G)$. Let E_0 be its weak*-closure. Then G has the Haagerup property if and only if $E_0 = B(G)$.
For $G = \mathbb{F}_2$, $B(G) = E_0 \neq E_p$.

Bizarre groups

So in general there seems to many large ideals in $B(G)$.

However, could the opposite be true, are there non-amenable groups G with no exotic large ideals in $B(G)$?
(Question raised by Tim De Laat, Copenhagen 2019.)
In particular can one have

$$B(G) = \mathbb{C}1 \oplus B_r(G)?$$

We call such a group *bizarre*.

Conjecture

There are no bizarre groups.

Crossed products

Start with an action α of a locally compact group G on a C^*-algebra B. We can then form the full crossed product

$$A_{full} = B \rtimes_\alpha G$$

and the reduced crossed product

$$A_{red} = B \rtimes_{\alpha,r} G,$$

but we should expect interesting algebras in between.

To make notation a little simpler we now assume that G is a discrete group.

Full and reduced crossed products

Then

$$A_{full} = B \rtimes_\alpha G = \overline{\text{span}}\{\pi(b)U_g \mid b \in B, g \in G\}$$

where π is a representation of B and U a unitary representation of G satisfying

$$U_g \pi(b) U_{g^{-1}} = \pi(\alpha_g(b))$$

with (π, U) universal.

If $B \subset B(\mathcal{H})$ we have

$$A_{red} = B \rtimes_{\alpha,r} G = \overline{\text{span}}\{\pi(b)\lambda(g) \mid b \in B, g \in G\}$$

where $\pi(b)$ and $\lambda(g)$ are operators on $L^2(G, \mathcal{H})$ given by

$$[\pi(b)f](x) = \alpha_{x^{-1}}(b)f(x) \text{ and}$$
$$[\lambda(g)f](x) = f(g^{-1}x).$$

Dual coactions

In both cases we have maps

$$\widehat{\alpha}_{red} : A_{red} \mapsto A_{red} \otimes_{min} C_r^*(G)$$

given by

$$\widehat{\alpha}_{red}[\pi(b)\lambda(g)] = (\pi(b) \otimes 1)(\lambda(g) \otimes \lambda_r(g))$$

and

$$\widehat{\alpha} : A_{full} \mapsto A_{full} \otimes_{max} C^*(G)$$

given by

$$\widehat{\alpha}[\pi(b)U_g] = (\pi(b) \otimes 1)(U_g \otimes g).$$

$\widehat{\alpha}_{red}$ is called the *dual (regular) coaction* of the regular crossed product and $\widehat{\alpha}$ is called the *dual R-coaction* of the full crossed product. (R is for Raeburn.)

Coactions for discrete groups

These are special cases of the following

Definition

A (regular) coaction of a C^*-algebra A is an injection $\delta : A \mapsto A \otimes_{min} C_r^*(G)$ satisfying

$$(\delta \otimes i) \circ \delta = (i \otimes \delta_r) \circ \delta$$

where $\delta_r(\lambda_r(g)) = \lambda_r(g) \otimes \lambda_r(g)$.

Definition

An R-coaction of a C^*-algebra A is an injection $\delta : A \mapsto A \otimes_{max} C^*(G)$ satisfying

$$(\delta \otimes i) \circ \delta = (i \otimes \delta_G) \circ \delta$$

where $\delta_G(g) = g \otimes g$.

(R is still for Raeburn.)

Very rough statement of Baum-Connes Conjecture

Conjecture (Baum-Connes)

If α is an action of a locally compact group G on a C^*-algebra B, then the K-theory of the reduced crossed product $B \rtimes_{\alpha,r} G$ is isomorphic to the "topological K-theory".

Problem

The topological K-theory is an exact functor of actions, but the reduced crossed product is not. (Gromov)

Crossed-Product Functors

Buss, Echterhoff, and Willett have studied certain properties of crossed-product functors.

A *crossed product* is a functor

$$A \mapsto A \rtimes_{\mu} G$$

from C^*-systems (A, G) to C^*-algebras together with natural transformations

$$A \rtimes_{\max} G \to A \rtimes_{\mu} G \to A \rtimes_{\mathrm{red}} G$$

restricting to the identity map on the dense subalgebra(s) $A \rtimes_{\mathrm{alg}} G$.

A crossed-product functor μ is *exact* if the sequence

$$0 \to I \rtimes_{\alpha,\mu} G \to A \rtimes_{\alpha,\mu} G \to B \rtimes_{\beta,\mu} G \to 0$$

is short exact whenever $0 \to (I, \alpha) \to (A, \alpha) \to (B, \beta) \to 0$ is short exact.
The full crossed product is exact, but the reduced crossed product is not, unless (by definition) G is exact.

A crossed-product functor μ is *Morita compatible* if roughly speaking

$$A \rtimes_{\alpha,\mu} G \sim_M B \rtimes_{\beta,\mu} G$$

whenever $(A, \alpha) \sim_M (B, \beta)$ equivariantly.
Both the full and reduced crossed products are Morita compatible.

BGW solution

Theorem (BGW)

There exists a unique minimal exact and Morita compatible crossed product functor ε.

Conjecture (BGW)

For any action (A, α, G), the ε-assembly map

$$\mu_\varepsilon : K_*^{top}(G; A) \to K_*(A \rtimes_{\alpha,\varepsilon} G)$$

is an isomorphism.

KLQ program

Require the crossed product $B \rtimes_{\alpha,\sigma} G$ to have a version δ^σ of the dual coaction $\widehat{\alpha}$, and find a coaction functor τ such that

$$
\begin{array}{ccc}
(B, \alpha) & \xmapsto{\text{full}} & (B \rtimes_\alpha G, \widehat{\alpha}) \\
& {\scriptstyle \sigma} \searrow & \Big\downarrow {\scriptstyle \tau} \\
& & (B \rtimes_{\alpha,\sigma} G, \delta^\sigma)
\end{array}
$$

commutes. Then do everything in terms of coaction functors.

So our philosophy is that to every crossed product functor there is a coaction functor making the above diagram commute.

Coaction Functors

Definition

A coaction functor is a functor

$$\tau : (A, \delta) \to (A^\tau, \delta^\tau)$$

on the catgory of coactions together with natural transformations satisfying ....

Theorem (KLQ)

There is a unique minimal, exact and Morita compatible coaction functor.

Let E be a large ideal in $B(G)$. The *E-ization coaction functor* $\tau_E\colon (A,\delta)\mapsto (A^E,\delta^E)$ is defined by letting A^E be the quotient of A by $\ker(\mathrm{id}\otimes q_E)\circ\delta$:

$$A \xrightarrow{\delta} A\otimes C^*(G) \xrightarrow{\mathrm{id}\otimes q_E} A\otimes C^*_E(G)$$

and letting δ^E and ϕ^E be the associated quotient maps.

Example

- $\tau_{B_r(G)}$ is normalization $(A,\delta)\mapsto (A^n,\delta^n)$
- $\tau_{B(G)}$ is the identity functor $(A,\delta)\mapsto (A,\delta)$
- (Maximalization is not τ_E for any E.)

For any action (A, α, G),

$$(A \rtimes_{\alpha, full} G)^E = A \rtimes_{\alpha, E} G \quad \text{and} \quad \widehat{\alpha}^E = \widehat{\alpha}_E.$$

This way we get an exotic crossed product functor

$$\mu_E = \tau_E \circ \text{full}.$$

comming from E by composing the full crossed product functor with τ_E.

Coaction properties of E-ization

Theorem

Every τ_E is Morita compatible.

Theorem

The coaction functor $\tau = \tau_E$ has the ideal property, i.e. given a coaction (A, δ) and a strongly δ-invariant ideal I, then

$$\iota^\tau : I^\tau \to A^\tau \text{ is injective.}$$

More coaction properties. (Exactness)

(The maximalization functor is exact.)

In general τ_E is not exact.

If τ is exact, then τ composed with the full crossed product will be an exact crossed-product functor.

Exact Large Ideals

Definition

A large ideal E is called *exact* if τ_E is exact.

Theorem (KLQ)

If E and F are exact large ideals, then so is $E \cap F$.
In particular $E \cap F = \langle EF \rangle$, the weak-closed linear span of the set EF of products.*

Example

Let E_p be the weak*-closure in $B(G)$ of $\text{span}\{P(G) \cap L^p(G)\}$, then for for every $p > 2$ and $G = \mathbb{F}_2$ we have

$$\langle E_p^2 \rangle \subset E_{p/2} \subsetneq E_p.$$

So E_p is not exact.

How to get more exact crossed products?

[Baum-Guentner-Willett]: Fix an action (C, γ) of G. For any action (B, α) of G, we get an action $\alpha \otimes \gamma$ on $B \otimes_{\max} C$ by $(\alpha \otimes \gamma)_g = \alpha_g \otimes \gamma_g$.
The homomorphism

$$\begin{aligned} B &\to B \otimes_{\max} C \qquad \text{given by} \\ b &\mapsto b \otimes 1 \qquad (\text{require } C \text{ unital}) \end{aligned}$$

is $\alpha - (\alpha \otimes \gamma)$ equivariant, so passes to the crossed products:

$$\phi : B \rtimes_\alpha G \to (B \otimes_{\max} C) \rtimes_{\alpha \otimes \gamma} G.$$

Definition (Baum-Guentner-Willett)

The *C-crossed product* is

$$B \rtimes_{\alpha, C} G = \phi(B \rtimes_\alpha G).$$

Has good properties

Theorem (Baum-Guentner-Willett)

The C-crossed product is an exact functor.

Theorem (Buss-Echterhoff-Willett)

There is a minimal C-crossed product given by

$$(C, \gamma) = (UCB_l(G), \textit{left translation}).$$

Motivation

Question (Baum-Guentner-Willett)

Maybe this is in fact the smallest exact crossed product functor?

(Still open.)

What is the corresponding coaction functor?

Main idea:

- Replace (C, γ) by a fixed coaction (D, ζ)
- Construct a coaction functor

$$(A, \delta) \mapsto (A^D, \delta^D)$$

such that

1. $(B \rtimes_\alpha G)^{C \rtimes_\gamma G} = B \rtimes_{\alpha, C} G$.
2. $(A, \delta) \mapsto (A^D, \delta^D)$ is an exact coaction functor.
3. The smallest coaction functor of this type is for

$$(D, \zeta) = (UCB_r(G) \rtimes_{\mathrm{rt}} G, \widehat{\mathrm{rt}}),$$

i.e., for every equivariant coaction (D, ζ) there is a canonical surjection

$$(A^D, \delta^D) \to (A^{UCB_r(G) \rtimes_{\mathrm{rt}} G}, \delta^{UCB_r(G) \rtimes_{\mathrm{rt}} G}).$$

We completed this case, mainly using Fell bundles.
Main idea:

- Replace (C, γ) by a fixed coaction (D, ζ)
- Construct a coaction functor

$$(A, \delta) \mapsto (A^D, \delta^D)$$

such that

$$B \rtimes_{\alpha,C} G = (B \rtimes_\alpha G)^D$$

when $(D, \zeta) = (C \rtimes_\gamma G, \widehat{\gamma})$.

Beginning of the construction

Since G is discrete, a coaction (A, δ) is just a *Fell-bundle structure* (or a graded algebra over G):
$A = \overline{\text{span}}_{s \in G} A_s$ where $A_s A_t \subset A_{st}$ and $A_s^* = A_{s^{-1}}$.

Definition

The *G-balanced tensor product* of coactions (A, δ) and (D, ζ) is the coaction $\delta \otimes_G \zeta$ given by the Fell bundle

$$\{A_s \otimes_{\max} D_s\}_{s \in G}$$

where for $A_s \otimes_{\max} D_s$ we take the closed span in $A \otimes_{\max} D$.
This gives a coaction on the C^*-algebra

$$A \otimes_G D := \overline{\operatorname*{span}_{s \in G}}(A_s \otimes_{\max} D_s) \subset A \otimes_{\max} D.$$

Finish the construction

We still need a homomorphism

$$(A, \delta) \to (A \otimes_G D, \delta \otimes_G \zeta).$$

For this, we further require the coaction (D, ζ) to be a *dual coaction* so

$$D = C \rtimes_\alpha G = \overline{\text{span}}\{\pi(c)V_g \mid c \in C, g \in G\}$$

where π is a representation of a *unital* C^*-algebra C and V a unitary representation of G satisfying

$$V_g \pi(b) V_{g^{-1}} = \pi(\alpha_g(b)).$$

Then $A_s \otimes_{\max} D_s = A_s \otimes_{\max} CV_s$ and we define the homomorphism on the fibers A_s by

$$a_s \mapsto a_s \otimes V_s.$$

It all works

Now define

$$A^D = \text{image of } A \text{ in } A \otimes_G D$$
$$\delta^D = \text{restriction of } \delta \otimes_G \zeta \text{ to } A^D.$$

Theorem (KLQ)

1. $(B \rtimes_\alpha G)^{C \rtimes_\gamma G} = B \rtimes_{\alpha,C} G$.
2. $(A, \delta) \mapsto (A^D, \delta^D)$ *is an exact coaction functor.*
3. *The smallest of this type is for*

$$(D, \zeta) = (\ell^\infty(G) \rtimes_{\mathrm{rt}} G, \widehat{\mathrm{rt}}),$$

i.e., for every equivariant coaction (D, ζ) *there is a canonical surjection*

$$(A^D, \delta^D) \to (A^{\ell^\infty(G) \rtimes_{\mathrm{rt}} G}, \delta^{\ell^\infty(G) \rtimes_{\mathrm{rt}} G}).$$

To-do list

Question

1. *Is this the minimal exact coaction functor?*
2. *What about locally compact G?*

General case — need a new plan

For G non-discrete, coactions are in general not given by Fell bundles, a coaction is a homomorphism

$$\delta : A \to \widetilde{M}(A \otimes C^*(G))$$

where unadorned $\otimes$ means minimal C^*-tensor product, and for any C^*-algebra C

$$\widetilde{M}(A \otimes C) = \{m \in M(A \otimes C) : m(1 \otimes C) \cup (1 \otimes C)m \subset A \otimes C\}.$$

New technique: promote the coactions so that they go into maximal (rather than minimal) tensor products:

$$\delta : A \to \widetilde{M}(A \otimes_{\max} C^*(G)).$$

(So maybe Raeburn was right all along. . .)

The plan

- Require $D = C \rtimes_\alpha G$ with $1 \in C$, so we have a $\delta_G - \zeta$ equivariant homomorphism $V : C^*(G) \to D$.
- Consider the composition

$$\begin{array}{ccc} A & \xrightarrow{\delta} & \widetilde{M}(A \otimes_{\max} C^*(G)) \\ & \searrow^{Q_D} & \downarrow \mathrm{id} \otimes V \\ & & \widetilde{M}(A \otimes_{\max} D) \end{array}$$

- Prove that the image $A^D = Q_D(A)$ carries a quotient δ^D of δ.
- Verify the 3 properties:
  1. recover C-crossed product,
  2. get an exact coaction functor, and
  3. identify smallest one.

Our result

Theorem (?(KLQ))

$A \to A^D$ is an exact coaction functor which recovers the C-crossed product. The smallest coaction functor of this type is for

$$(D, \zeta) = (UCB_r(G) \rtimes_{\mathrm{rt}} G, \widehat{\mathrm{rt}}).$$

At least we belive it is true, but the general case is not so smooth, we have to use exactness of the C-crossed product to show that our D-coaction is exact, while we wanted it the other way as in the discrete case.

E-ization vs Tensor-D

Assume G discrete. We have $D = C \rtimes_\alpha G$ so there is a map $V : C^*(G) \to D$, and $E = \ker V^\perp$ is a large ideal with $V(C^*(G)) = C^*_E(G)$.

E-ization (R-version) sends A into $A \otimes_{\max} C^*_E(G)$.
while Tensor-D sends A into
$A \otimes_G D := \overline{\mathrm{span}}_{s \in G}(A_s \otimes_{\max} D_s) \subset A \otimes_{\max} D$.

In both cases $a_s \to a_s \otimes V_s$ for $a_s \in A_s$.
Let ι be the inclusion of $C^*_E(G)$ into D. Then we get a map

$$id \otimes \iota : A \otimes_{\max} C^*_E(G) \to A \otimes_{\max} D.$$

Problem

$id \otimes \iota$ is not injective in general, since we are in $\otimes_{\max}$-land.

G is amenable at infinity

Theorem (BEW)

If G is amenable at infinity then the crossed-product functor $(A,\alpha) \mapsto A \rtimes_{\alpha,\mathrm{UCB_r}(G)} G$ is naturally isomorphic to the reduced crossed product $(A,\alpha) \mapsto A \rtimes_{\alpha,r} G$, and is consequently strictly smaller than the full crossed product functor.

We have seen that we needed two kinds of coactions:
Regular coactions:

$$\delta : A \to \widetilde{M}(A \otimes_{\min} C^*(G)).$$

and R-coactions:

$$\delta : A \to \widetilde{M}(A \otimes_{\max} C^*(G)).$$

Maybe we also need C^*-algebras between $A \otimes_{\max} C^*(G)$ and $A \otimes_{\min} C^*(G)$, or in general C^*-algebras between $A \otimes_{\max} B$ and $A \otimes_{\min} B$.
So out there is probably an unexplored forest of exotic tensor products waiting to be discovered.

More questions

Other objects worth studying are the algebras

$$UCB_r(G) \rtimes_{\mathrm{rt}} G$$

which for G discrete are the Roe-algebras

$$\ell^\infty(G) \rtimes_{\mathrm{rt}} G.$$

Brown-Guentner approach

- Start with a G-invariant subspace D of $B(G)$.
- Consider *D-representations* of G, i.e., representations for which a dense set of vectors gives coefficient functions in D.
- Form a quotient $C^*_{D,BG}(G)$ of $C^*(G)$ by the intersection of the kernels of all D-representations.

(fine print: we need the weak* closure of D to contain $B_r(G)$.)

Examples

Example

If $D = B(G)$, then $C^*_{D,BG}(G) = C^*(G)$.

Example

If $D = C_c(G) \cap B(G)$ or $L^p(G) \cap B(G)$ for $1 \leq p \leq 2$, then $C^*_{D,BG}(G) = C^*_r(G)$.

Example

(Okayasu-Higson-Ozawa)
If $G = \mathbb{F}_n$ then $D_p = \ell^p(G) \cap B(G)$ for $2 < p < \infty$ gives a continuum of pair-wise distinct quotients $C^*_{D_p,BG}(G)$.

Compare with our approach

Again start with a G-invariant subspace D of $B(G)$.

- The preannihilator ${}^{\perp}D$ is an ideal of $C^*(G)$ (by G-invariance).
- The quotient $C^*_{D,KLQ}(G) := C^*(G)/{}^{\perp}D$ is an exotic group C^*-algebra.

Clearly $C^*_{D,BG}(G) = C^*_{D,KLQ}(G)$, right?

The $P(G)$ question

This leads to:

Theorem (BEW)

If G is a locally compact group, D a nonzero G-invariant ideal of $B(G)$, and $D = \operatorname{span}\{D \cap P(G)\}$, then $C^*_{D,BG}(G) = C^*_{D,KLQ}(G)$.

Again, for our purposes it would be enough to know that $\operatorname{span}\{D \cap P(G)\}$ is weak* dense in D.

But, somehow surprising, the question as stated seems to be open — at least to nonexperts like us.

Partial results

Proposition (Buss-Echterhoff-Willett, KLQ)

The $P(G)$ question has a positive answer if D is norm closed in $B(G)$, so in particular for $D = C_0(G) \cap B(G)$.

By far the most important case of the $P(G)$ question that still eludes us is:

Question (The L^p question)

For $2 < p < \infty$, is

$$L^p(G) \cap B(G) = \mathrm{span}\{L^p(G) \cap P(G)\}?$$

Surely someone must know the answer to this?